Math 421-002 **In-class Worksheet #1** **February 12, 2020**

Definition. A **metric** on a space X is given by a function $d : X \times X \to \mathbb{R}$, which satisfies all of the following conditions:

1. $\forall x, y \in X, d(x, y) \geq 0$,
2. $d(x, y) = 0$ if and only if $x = y$,
3. $\forall x, y \in X, d(x, y) = d(y, x)$,
4. $\forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z)$.

Problem 1. Show that any norm defines a metric by $d(x, y) = ||x - y||$.

The **discrete metric** ρ on a space X is defined by

$$\rho(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Problem 2. Prove that ρ is indeed a metric (i.e. satisfies conditions 1-4 above).

Problem 3. Does any metric define a norm?

Definition. Given a space X with metric d, an **open ball** centered at x of radius ε is the set $B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$.

Problem 4. Given a space X with the discrete metric ρ, prove the following statements.

(a) For any $x \in X$, the set $\{x\}$ is an open set.

(b) All sets in X are open.

(c) All sets in X are closed.

(d) Any subset $A \subseteq X$, such that $|A| \geq 2$ is disconnected.

Note: (X, ρ) has the largest possible topology, i.e., all subsets of X are open. This is also referred to as *discrete topology.*

Problem 5. Consider the set $E = (2, 3) \cup [4, 5] \cup [6, \infty) \subseteq \mathbb{R}$.

(a) Prove that $(2, 3)$ is both relatively open and relatively closed in E and $[4, 5]$ is also both relatively open and relatively closed in E.

(b) Prove that E is disconnected.